Assignment 2

This homework is due *Tuesday* Sep 20.

There are total 22 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 2.1 (up to Thm 2.1.4) in Bartle–Sherbert.

1. Quick cheat-sheet.

Reminder. (Subsection 2.1.1) On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$,
(A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$,
(A3) there exists $0 \in \mathbb{R}$ s.t. $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$,
(A4) for each $a \in \mathbb{R}$ there exists an element $-a$ s.t. $a + (-a) = (-a) + a = 0$,
(M1) $ab = ba$ for all $a, b \in \mathbb{R}$,
(M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$,
(M3) there exists $1 \in \mathbb{R}$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$,
(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a \cdot (1/a) = (1/a) \cdot a = 1$,
(D) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{R}$.

Note that within this homework, we adopt textbook's viewpoint on rational numbers, that is that they are a specific subset of $\mathbb{R}$.

2. Exercises.

(1) (Parts of 2.1.1, 2, 5) For $a, b \in \mathbb{R}$, prove that
  (a) [1pt] $-(a + b) = -a + (-b)$,
  (b) [1pt] $-(-a) = a$,
  (c) [1pt] $-(a/b) = (-a)/b$ if $b \neq 0$,
  (d) [1pt] if $a \neq 0$, $b \neq 0$, then $1/(ab) = (1/a)(1/b)$.

  Every equality you write should be accompanied by a reference to the exact property of real numbers or theorem you are using.

(2) [1pt] Prove that if $a \in \mathbb{R}$ satisfies $a \cdot a = a$, then $a = 0$ or $a = 1$.

(3) (a) [1pt] Is addition of real numbers distributive over multiplication?
  (b) [1pt] Is set union distributive over set intersection? That is, is it true that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, C?
  (c) [1pt] Is set intersection distributive over set union?

— *see next page* —

(4) (2.3.3ab) Solve the following equations, justifying each step by referring to an appropriate property or theorem.
   (a) [1pt] $2x + 5 = 8$,
   (b) [1pt] $x^2 = 2x$.

(5) On the set $\mathbb{N}$, consider two operations: $\oplus$ and $\odot$ defined as follows: $a \oplus b = ab$ and $a \odot b = a^b$.
   (a) [1pt] Do properties A1, A2 hold for $\oplus$? That is, is it true that $a \oplus b = b \oplus a$, and that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in \mathbb{N}$?
   (*Hint:* For this and further items, the main way to figure out questions is to write out expressions with $\odot$ and $\oplus$ in terms of "usual" operations, using definition of $\odot$ and $\oplus$.)
   (b) [1pt] Do properties M1, M2 hold for $\odot$?
   (c) [1pt] Is there unit element with respect to $\odot$? That is, is there an element $1_\odot \in \mathbb{N}$ such that $1_\odot \odot a = a \odot 1_\odot = a$ for all $a \in \mathbb{N}$?
   (d) [1pt] Is there a *left* unit element with respect to $\odot$? That is, is there an element $1_\ell \in \mathbb{N}$ such that $1_\ell \odot a = a$ for all $a \in \mathbb{N}$?
   (e) [1pt] Is $\odot$ distributive over $\oplus$ *on the left*? That is, is it true that $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$?
   (f) [1pt] Is $\odot$ distributive over $\oplus$ *on the right*? That is, is it true that $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot a)$?

(6) (a) [1pt] (Ex. 2.1.8a) Let x, y be rational numbers. Prove that $xy, x + y$ are rational numbers.
   (b) [1pt] (Ex. 2.1.8b) Let x be a rational number, y an irrational number. Prove that $x + y$ is irrational. Prove that if, additionally, $x \neq 0$, then xy is irrational.
   (c) [1pt]Let x, y be irrational numbers. Is it true that $x + y$ is always irrational? Is it true that $x + y$ is always rational?
   (d) [1pt] Same two questions for xy.

(7) (a) [1pt] Prove that there does not exist a rational number r such that $r^2 = 3$.
   (b) [1pt] Prove that there does not exist a rational number r such that $r^2 = 27$.